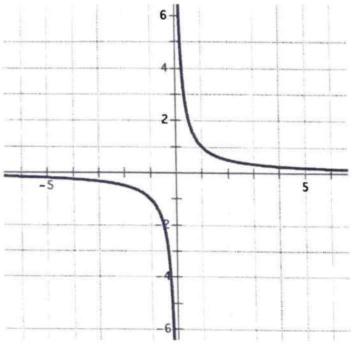
 Name: KEY

Horizontal and Vertical Asymptotes

Horizontal Asymptotes:

Graph of $f(x)=\frac{1}{x}$



$\lim_{x\to\infty}\frac{1}{x}=$ 0 $\lim_{x\to-\infty}\frac{1}{x}=$ 0

Limits as x goes to INFINITY will create a horizontal asymptote!

The horizontal line $y=0$ is a horizontal asymptote of the graph of the function if either $\lim_{x\to\infty} f(x)=0$ or $\lim_{x\to-\infty} f(x)=0$. Horizontal asymptotes show the end behavior of a function as x approaches $\pm\infty$.

Since there are two ways to create a horizontal asymptote, a graph can have up to two (but this is unusual- most will only have one, and all rational functions only have one).

RULES (these should look familiar):

For a rational function (a polynomial function over a polynomial function):

- If the degree of the denominator is bigger then there is a horizontal asymptote at $y=0$.
- If the degree of the numerator is bigger then there is *slant* asymptote.
- If the degrees are equal, then there is horizontal asymptote at $y=\frac{a_n}{b_n}$, where a_n and b_n are the leading coefficients of the numerator and the denominator.

Limits involving Infinity

Why does this work?

Algebraically determine the $\lim\limits_{x\to\infty}\dfrac{5x^2-3x+1}{2x^2+4x-7}$

Divide every term by the *highest* power in the denominator:

The limit can now be expressed as $\lim\limits_{x\to\infty}\dfrac{\frac{5x^2}{x^2}-\frac{3x}{x^2}+\frac{1}{x^2}}{\frac{2x^2}{x^2}+\frac{4x}{x^2}-\frac{7}{x^2}} = \lim\limits_{x\to\infty}\dfrac{5-\frac{3}{x}+\frac{1}{x^2}}{2+\frac{4}{x}-\frac{7}{x^2}} = \dfrac{5-0+0}{2+0-0} = \dfrac{5}{2}$

For non-rational functions (exponential, logarithmic, trig), we use the graph (visual limits) to determine limits/asymptotes or we may intuitively know. We'll do this on another day!

Relating Horizontal Asymptotes to Limits

Determine the value of the limits:

$\lim\limits_{x\to\infty}\dfrac{3x^3-x+1}{x+3} = \lim\limits_{x\to\infty}\dfrac{3x^2-1+\frac{1}{x}}{1+\frac{3}{x}} = \infty$	$\lim\limits_{x\to\infty}\dfrac{1-7x^2}{x+5} = \lim\limits_{x\to\infty}\dfrac{\frac{1}{x}-7x}{1+\frac{5}{x}} = -\infty$
$\lim\limits_{x\to\infty}\dfrac{7x^2+1}{-x+5} = \lim\limits_{x\to 0}\dfrac{-7x-\frac{1}{x}}{1-\frac{5}{x}} = -\infty$	$\lim\limits_{x\to\infty}\dfrac{5x}{x^2+1} = 0$ (horiz. asymp)
$\lim\limits_{x\to\infty}\dfrac{3x^2}{2x^2+1} = \dfrac{3}{2}$	$\lim\limits_{x\to\infty}\dfrac{3x}{x^3-1} = 0$
$\lim\limits_{x\to-\infty}\dfrac{8x+1}{x^2} = 0$	$\lim\limits_{x\to\infty}\dfrac{-8x+1}{x^2} = 0$

Limits involving Infinity

Vertical Asymptotes:

You already know that if you want to find a vertical asymptote in a rational function, look for values that would make denominator equal to ZERO **but** ones that don't cancel with a term from the numerator (when a term cancels out from the numerator and denominator then it makes a HOLE). So here is how it looks with limits:

- ➔ If $\lim_{x\to a} f(x) = \frac{0}{0}$ then there is a HOLE at x = a
- ➔ If $\lim_{x\to a} f(x) = \frac{n}{0}$, where n is a real number, then there is a vertical asymptote at x = a

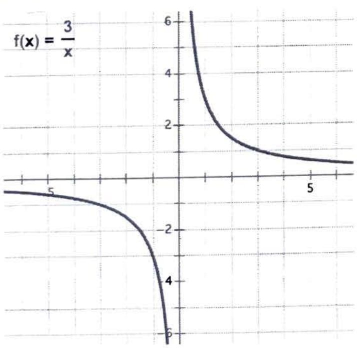
The line $x = a$ is a vertical asymptote of the graph of $f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty$$

$\lim_{x\to a^+}$ means you are looking at the RIGHT side of the x-value "a"

$\lim_{x\to a^-}$ means that you are looking at the LEFT side of the x-value "a"

Visually:



$\lim_{x\to 0^+} \frac{3}{x} = \infty$

$\lim_{x\to 0^-} \frac{3}{x} = -\infty$

Therefore, $\lim_{x\to 0} \frac{3}{x} =$ DNE

Limits involving Infinity

Example: Determine the value of $\lim\limits_{x \to -3} \frac{x^2+1}{3+x}$ if it exists.

Must first consider $\lim\limits_{x \to -3^-} \frac{x^2+1}{3+x} = \frac{+}{-} = -\infty$ and $\lim\limits_{x \to -3^+} \frac{x^2+1}{3+x} = \frac{+}{+} = \infty$

Therefore, $\lim\limits_{x \to -3} \frac{x^2+1}{3+x} =$ DNE

b/c the one-sided limits don't match!

Examples:

Find the following limits, but be sure to check both the left and right sides!

$\lim\limits_{x \to 1} \frac{4}{x-1}$

$\lim\limits_{x \to 1^+} \frac{4}{x-1} = \frac{+}{+} = \infty$

$\lim\limits_{x \to 1^-} \frac{4}{x-1} = \frac{+}{-} = -\infty$

so $\lim\limits_{x \to 1} \frac{4}{x-1} =$ DNE!

$\lim\limits_{x \to 1} \frac{2}{(x-1)^2}$

$\lim\limits_{x \to 1^+} \frac{2}{(x-1)^2} = \frac{+}{+} = \infty$

$\lim\limits_{x \to 1^-} \frac{2}{(x-1)^2} = \frac{+}{+} = \infty$

so $\lim\limits_{x \to 1} \frac{2}{(x-1)^2} = \infty$

$\lim\limits_{x \to 0} \frac{\sqrt{1-x}}{x^2}$

$\lim\limits_{x \to 0^+} \frac{\sqrt{1-x}}{x^2} = \frac{+}{+} = \infty$

$\lim\limits_{x \to 0^-} \frac{\sqrt{1-x}}{x^2} = \frac{+}{+} = \infty$

so $\lim\limits_{x \to 0} \frac{\sqrt{1-x}}{x^2} = \infty$

$\lim\limits_{x \to 5} \frac{x^2+5x}{x^2-25}$

$\lim\limits_{x \to 5^+} \frac{x^2+5x}{x^2-25} = \frac{+}{+} = \infty$

$\lim\limits_{x \to 5^-} \frac{x^2+5x}{x^2-25} = \frac{+}{-} = -\infty$

so $\lim\limits_{x \to 0} \frac{x^2+5x}{x^2-25} =$ DNE